

Homework

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HW

due 2025-02-11, before class

Exams (tentatively)

midterm : **Thursday, March 27, 2025, 12:30-13:45**
final: **tbd**

V. CENTRAL LIMIT THEOREM [(2+2+3+4) PTS]

Let x be a random variable with (probability) distribution function $w(x)$, which is positive, $w(x) \geq 0$, and normalized $\int_{-\infty}^{\infty} w(x) dx = 1$. The average, \bar{x} , and deviation, $(\Delta x)^2$, are defined as

$$\bar{x} \equiv \int_{-\infty}^{\infty} x w(x) dx$$

$$(\Delta x)^2 \equiv \int_{-\infty}^{\infty} (x - \bar{x})^2 w(x) dx = \overline{x^2} - \bar{x}^2.$$

We assume that both these quantities exist (for, e.g., $w(x) = (1 + x^2)^{-1}/\pi$ the above integrals do not converge).

- a) Let us assume that the random numbers are the result of a measurement of one observable. It makes sense to calculate the arithmetic average, $y = \frac{1}{N} \sum_{i=1}^N x_i$, of N measurements, x_i . We can now assume that y is another random variable with the distribution function $w_N(y)$. The latter is obtained by summation of the probabilities $w(x_1) \cdots w(x_N) dx_1 \cdots dx_N$ with the condition that $y = \frac{1}{N} \sum_{i=1}^N x_i$, i.e.

$$w_N(y) = \int dx_1 \cdots dx_N w(x_1) \cdots w(x_N) \delta \left(y - \frac{1}{N} \sum_{i=1}^N x_i \right).$$

Calculate the average and the deviation of y and discuss the results.

- b) Calculate the average and the deviation of

$$u \equiv \frac{1}{\sqrt{N} \Delta x} \sum_{i=1}^N (x_i - \bar{x}).$$

- c) The Fourier transform $\tilde{w}(t) \equiv \int e^{itx} w(x) dx = \overline{e^{itx}}$ of the distribution function $w(x)$ is a generating function of the "moments" $\overline{x^\kappa} = \int x^\kappa w(x) dx$ of the distribution, in the following sense:

$$\overline{x^\kappa} = \left(\frac{1}{i} \frac{d}{dt} \right)^\kappa \tilde{w}(t) \Big|_{t=0}, \quad \kappa = 1, 2, 3, \dots$$

Show this identity. Conversely, these moments also define the distribution function $w(x)$. Find the distribution function for the moments defined as $\overline{x^\kappa} = \kappa! a^{-\kappa}$ for even κ and $\overline{x^\kappa} = 0$ for odd κ (a is a positive constant).

★ Use the integral $\frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{-itx} \frac{a^2}{a^2 + t^2} = \frac{a}{2} e^{-a|x|}$ (which can be calculated using residuals).

- d) The central limit theorem states that – under rather general conditions for $w(x)$ – the quantity u [see b)] follows the Gaussian (or normal) distribution

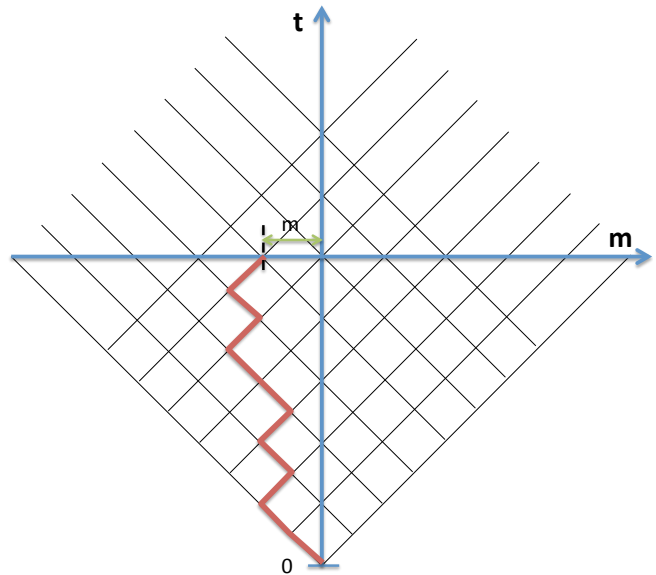
$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}$$

for $N \rightarrow \infty$. To prove this, use the generating function $\tilde{w}_N(t) = \int e^{itu} w_N(u) du$ [u and the corresponding $w_N(u)$ as in b)]. Show that $\tilde{w}_N(t) \propto \left(\tilde{w}(t/(\Delta x \sqrt{N})) \right)^N$, where $\tilde{w}(t)$ is the Fourier transform of $w(x)$. Then expand, $\ln \tilde{w}_N(t)$, to second order in t . The normal distribution follows from the inverse Fourier transform.

VI. DIRECTED POLYMER [(2+3) PTS]

Let us consider a square grid, which has been rotated by 45^{deg} (see figure). On this grid we define a path, the directed polymer, which starts at the origin ($t = 0$) and extends in the positive t -direction (at each grid point the path goes either left or right; and steps in the negative t -direction are not permitted).

All paths of length N end at the same t -position (depicted by the m -axis in the figure). N is therefore the number of connected cell edges constituting the polymer. The distance of the endpoint to the t -axis is characterized by a number m . Each path represents a microstate and the endpoint (i.e. m) the macrostate.



- Find the number of microstates, $W(m)$, for each macrostate, m . How many microstates exist for a directed polymer of length N ?
- What is the value m_0 of m for which $W(m)$ is maximal? Expand $\ln(W(m))$ around m_0 ($m = m_0 + \Delta m$) to second order in Δm .

Hint: You can assume that the grid points of the rotated grid have integer m coordinates and that $N + m$ is even.