Homework

HW 6

due *2024-04-25*

Exams	final: **Tuesday, May 7, 2024**, *10:00-12:00*

XXII. IDEAL FERMIONS AT LOW TEMPERATURES AND SOMMERFELD EXPANSION *[(5+9+3) PTS]*

Here you should apply the Sommerfeld expansion to calculate properties of Fermions at low temperatures (see lecture).

a) Calculate $E(T, N, V)$ up to order $\mathcal{O}(T^2)$ for a given single particle density of states (DoS) $\rho(\epsilon)$. Derive from that the result for the specific heat (in order $\mathcal{O}(T)$):

$$C_V(T, N, V) = \left(\frac{\partial E}{\partial T}\right)_{V,N} = \frac{\partial E}{\partial T} + \frac{\partial E}{\partial \mu}\left(\frac{\partial \mu}{\partial T}\right)_{V,N}$$

b) Derive an expression for the single particle DoS $\rho(\epsilon)$ in d dimensions for non-relativistic Fermions, with energy-momentum relation $\epsilon = \mathbf{p}^2/(2m)$. What is special for $d = 2$? Use this expression to calculate the energy, E, (order $\mathcal{O}(T^2)$), heat capacity, C_V, (order $\mathcal{O}(T)$), and pressure, $p(T, N, V) = -(\partial E/\partial V)_{N,T}$, (order $\mathcal{O}(T^2)$) - all for arbitrary dimension d.

c) Calculate the Fermi-energy and heat capacity for relativistic Fermions with linear energy-momentum relation, $\epsilon_{\mathbf{p}} = c|\mathbf{p}|$ (e.g., electrons in a white dwarf star) for arbitrary dimension d.

XXIII. BOSE-EINSTEIN CONDENSATION *[(4+6+8) PTS]*

The grand canonical potential for an ideal (non-relativistic) Bose gas is given by

$$J(T,\mu,V) = T\ln(1-z) - T\frac{V}{\lambda_\beta^3}\mathrm{Li}_{5/2}(z)\,,$$

where $z = e^{\mu/T}$, $\lambda_\beta = h/\sqrt{2\pi m T}$, and $\mathrm{Li}_s(z)$ is the polylogarithm function.

a) Use the potential J and calculate the entropy $S(T,V,\mu) = -(\frac{\partial J}{\partial T})_{V,\mu}$ above and below the transition point of the Bose condensation. Show that S is continuous at the transition point and vanishes for $V=0$ and $T=0$.

b) Calculate the specific heat

$$C_V(T,V,z) = T\left(\frac{\partial S}{\partial T}\right)_{V,N}$$

above and below the transition. In the following the volume is fixed. Notice, that the partial derivative of $S(T,V,\mu)$ needs to be calculated while keeping N constant, i.e.,

$$\left(\frac{\partial S}{\partial T}\right)_{V,N} = \left(\frac{\partial S}{\partial T}\right)_z + \frac{\partial S}{\partial z}\left(\frac{\partial z}{\partial T}\right)_N\,.$$

The derivative $\partial z/\partial T$ is obtained by applying the operator $(\partial/\partial T)_N$ on both sides of

$$N(T,V,\mu) = -\left(\frac{\partial J}{\partial \mu}\right)_{T,V} = \frac{z}{1-z} + \frac{V}{\lambda_\beta^3}\mathrm{Li}_{3/2}(z)\,. \tag{1}$$

Is the specific heat continuous at the transition point?

c) Calculate the inverse isothermal compressibility

$$K_T^{-1} = -V\left(\frac{\partial p}{\partial V}\right)_{T,N}$$

above and below the transition. Show that K_T diverges at the transition point. Notice again, that the derivative is to be calculated for constant N (T is fixed):

$$p \approx \frac{T}{\lambda_\beta^3}\mathrm{Li}_{5/2}(z) \Rightarrow \left(\frac{\partial p}{\partial V}\right)_N = \left(\frac{\partial p}{\partial V}\right)_z + \frac{\partial p}{\partial z}\left(\frac{\partial z}{\partial V}\right)_N\,.$$

In order to calculate $(\frac{\partial V}{\partial z})_{T,N}$ solve eq. (1) for V. Use the above results to calculate K_T as function of the specific volume $v = V/N$ near the critical volume $v_c(T) = \lambda_\beta^3/\mathrm{Li}_{3/2}(1)$ at which the Bose condensation occurs, i.e., for $v - v_c \ll v_c$. Use the series expansion

$$\mathrm{Li}_{3/2}(z) = \mathrm{Li}_{3/2}(1)\left(1 - 1.36\sqrt{1-z} + \ldots\right)\,.$$

If you calculated all correctly, S is continuous at the transition point and vanishes at $v=0$ (or $T=0$). Furthermore, $C_V(T,V,z)$ has a jump in the first derivative and K_T diverges at the transition point. Therefore, the Bose-Einstein condensation (BEC) is a phase transition of second order, if one considers the mixed phase of gas and condensate in the region $v < v_c$ as a new phase. On the other hand, if the new phase is only the phase when all particles are in the condensate at $T=0$ (this state corresponds to the specific volume $v=0$, since there is no repulsion), the BEC is a phase transition of first order for all isotherms.

You can submit 2 of the following 3 optional problems!

XXIV. ISING SPIN SYSTEM IN MAGNETIC FIELD (OPTIONAL) *[(2+2+4+4) PTS]*

In a classical Ising spin system with N spins, each spin can be in one of two states: $S = \pm 1$. We consider such a system of non-interacting spins in an external magnetic field H. The energy of a single spin i has the value $E_i = \mu H S_i = \epsilon_0 S_i$. The macrostates are defined by the total energy $E = M\epsilon_0$ for $M = -N, \ldots, N$. Calculate:

a) the number of microstates in a macrostate,

b) the entropy $S_B(E)$,

c) the temperature $T(E)$ for $N \gg 1$, and

d) energy and entropy as function of temperature.

XXV. TWO-COMPONENT GAS (OPTIONAL) *[(3+2+2) PTS]*

Here we consider a gas consisting of two types of particles, which only differ in their mass (e.g. two isotopes of one element). N_1 particles have the mass m_1 and N_2 particles mass m_2. All particles should be considered classical and non-interacting, i.e.,

$$\mathcal{H}(\vec{X}) = \sum_{i=1}^{N_1} \frac{p_i^2}{2m_1} + \sum_{i=N_1+1}^{N_1+N_2} \frac{p_i^2}{2m_2}.$$

The gas has temperature T and is enclosed in volume V.

a) Calculate the Helmholtz free energy $F(T, V, N_1, N_2)$. Verify the homogeneity relation $F(T, \lambda V, \lambda N_1, \lambda N_2) = \lambda F(T, V, N_1, N_2)$.

b) Calculate from that the Gibbs free energy $G(T, p, N_1, N_2)$ via Legendre transformation. What is the homogeneity relation for G?

c) From the homogeneity relation for G derive the Gibbs-Duhem relation $G(T, p, N_1, N_2) = \mu_1 N_1 + \mu_2 N_2$.

XXVI. IDEAL GAS OF RELATIVISTIC PARTICLES (OPTIONAL) *[4+3 PTS]*

We consider an ideal gas of N non-interacting relativistic particles in a volume $V = L^3$ in 3 dimensions. These particles follow the Boltzmann statistics. The energy of a particle i is proportional to its momentum, i.e., $\epsilon_i = c|\mathbf{p}_i|$.

a) Calculate $E(\beta)$ and the heat capacity C of the gas.

a) Calculate $S_B(\beta)$.