Homework HW 2

due *2024-02-15*, before class

Exams (tentatively)	midterm : **Tuesday, March 19, 2024**, *11:00-12:15* final: **Tuesday, May 7, 2024**

V. CENTRAL LIMIT THEOREM *[(2+2+3+4) PTS]*

Let x be a random variable with (probability) distribution function $w(x)$, which is positive, $w(x) \geq 0$, and normalized $\int\limits_{-\infty}^{\infty} w(x)\,dx = 1$. The average, $\overline{x}$, and deviation, $(\Delta x)^2$, are defined as

$$\overline{x} \equiv \int\limits_{-\infty}^{\infty} x w(x)\,dx$$

$$(\Delta x)^2 \equiv \int\limits_{-\infty}^{\infty} (x - \bar{x})^2 w(x)\,dx = \overline{x^2} - \overline{x}^2\,.$$

We assume that both these quantities exist (for, e.g., $w(x) = (1 + x^2)^{-1}/\pi$ the above integrals do not converge).

a) Let us assume that the random numbers are the result of a measurement of one observable. It makes sense to calculate the arithmetic average, $y = \frac{1}{N}\sum_{i=1}^{N} x_i$, of N measurements, x_i. We can now assume that y is another random variable with the distribution function $w_N(y)$. The latter is obtained by summation of the probabilities $w(x_1)\cdots w(x_N)\,dx_1\cdots dx_N$ with the condition that $y = \frac{1}{N}\sum_{i=1}^{N} x_i$, i.e.

$$w_N(y) = \int dx_1 \cdots dx_N\, w(x_1)\cdots w(x_N)\,\delta\left(y - \frac{1}{N}\sum_{i=1}^{N} x_i\right)\,.$$

Calculate the average and the deviation of y and discuss the results.

b) Calculate the average and the deviation of

$$u \equiv \frac{1}{\sqrt{N}\Delta x}\sum_{i=1}^{N}(x_i - \overline{x})\,.$$

c) The Fourier transform $\tilde{w}(t) \equiv \int e^{itx} w(x)\,dx = \overline{e^{itx}}$ of the distribution function $w(x)$ is a generating function of the "moments" $\overline{x^\kappa} = \int x^\kappa w(x)\,dx$ of the distribution, in the following sense:

$$\overline{x^\kappa} = \left(\frac{1}{i}\frac{d}{dt}\right)^\kappa \tilde{w}(t)\big|_{t=0}\,,\ \ \kappa = 1, 2, 3, \ldots\,.$$

Show this identity. Conversely, these moments also define the distribution function $w(x)$. Find the distribution function for the moments defined as $\overline{x^\kappa} = \kappa! a^{-\kappa}$ for even κ and $\overline{x^\kappa} = 0$ for odd κ (a is a positive constant).
★ Use the integral $\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{-itx}\frac{a^2}{a^2+t^2} = \frac{a}{2}e^{-a|x|}$ (which can be calculated using residuals).

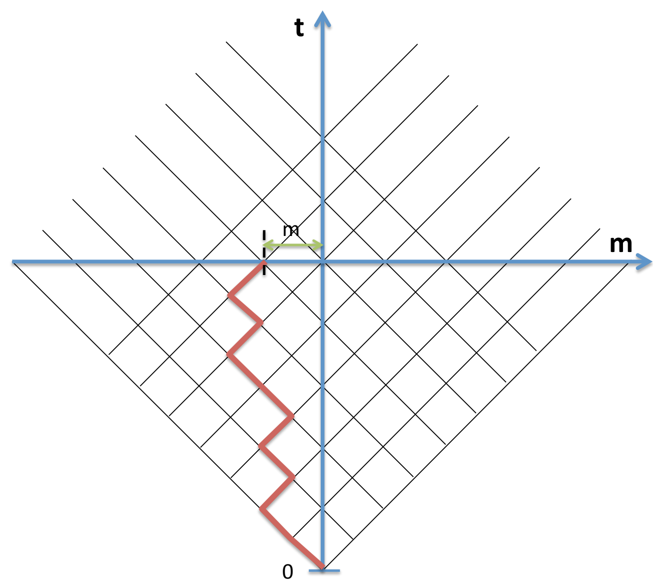
d) The central limit theorem states that – under rather general conditions for $w(x)$ – the quantity u [see b)] follows the Gaussian (or normal) distribution

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}$$

for $N \to \infty$. To prove this, use the generating function $\tilde{w}_N(t) = \int e^{itu} w_N(u)\, du$ [u and the corresponding $w_N(u)$ as in b)]. Show that $\tilde{w}_N(t) \propto \left(\tilde{w}(t/(\Delta x\sqrt{N}))\right)^N$, where $\tilde{w}(t)$ is the Fourier transform of $w(x)$. Then expand, $\ln \tilde{w}_N(t)$, to second order in t. The normal distribution follows from the inverse Fourier transform.

VI. DIRECTED POLYMER *[(2+3) PTS]*

Let us consider a square grid, which has been rotated by $45^{\deg}$ (see figure). On this grid we define a path, the directed polymer, which starts at the origin ($t = 0$) and extends in the positive t-direction (at each grid point the path goes either left or right; and steps in the negative t-direction are not permitted).
All paths of length N end at the same t-position (depicted by the m-axis in the figure). N is therefore the number of connected cell edges constituting the polymer. The distance of the endpoint to the t-axis is characterized by a number m. Each path represents a microstate and the endpoint (i.e. m) the macrostate.



a) Find the number of microstates, $W(m)$, for each macrostate, m. How many microstates exist for a directed polymer of length N?

b) What is the value m_0 of m for which $W(m)$ is maximal? Expand $\ln(W(m))$ around m_0 ($m = m_0 + \Delta m$) to second order in Δm.

Hint: You can assume that the grid points of the rotated grid have integer m coordinates and that $N + m$ is even.