# Warm-up homework

HW 1

**due** *2024-01-30*, before class

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| Exams (tentatively) | midterm : | **Tuesday, March 19, 2024**, *11:00-12:15* |
| | final: | **Tuesday, May 7, 2024** |

## I. GAMMA FUNCTION AND STIRLING'S FORMULA *[(3+5) PTS]*

a) The Gamma function is defined as $\Gamma(z+1) = \int_0^\infty dt\, e^{-t} t^z$ for $\mathrm{Re}(z) > -1$. It satisfies the functional equation $\Gamma(z+1) = z\Gamma(z)$ with $\Gamma(1) = 1$. Therefore $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$. Prove the functional equation for real arguments by partial integration.

b) Derive *Stirling's formula* for large $n$

$$\Gamma(n+1) = n! \approx \sqrt{2\pi n} n^n e^{-n}$$

by expanding the exponent in the integrand of

$$\Gamma(n+1) = \int_0^\infty dt\, e^{-t} t^n = \int_0^\infty dt\, e^{-t+n\ln t}$$

around its maximum to second order in $t$ (saddle point approximation). How does the width of the maximum depend on $n$?
★ "Decrease" the lower integration limit to $-\infty$ (Justify!); $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

## II. VOLUME AND SURFACE AREAS IN HIGH-DIMENSIONAL SPACES *[(3+1+3+4+3) PTS]*

An important (and maybe surprising) fact for statistical physics is that
▹ ***a*lmost the entire volume of a high-dimensional body is located just below its surface**.
Since we have applications in phase-space in mind, the spacial dimension, $D$, is on the order of the Avogadro number ($N_A \approx 6.022 \times 10^{23}$).

a) As a start, we can check this by simple dimensional analysis. To this end, we assume that the volume $V_D(L)$ of the $D$-dimensional body can be written as $V_D(L) = L^D V_D(1)$, where $L$ is the typical (linear) length of the body. $V_D(1)$ is assumed to depend only on the aspect ratios of the length scales and angles, which describe the shape of the body and is invariant with respect to compression or expansion. Show that for arbitrary, fixed $\kappa$

$$\lim_{D\to\infty} \frac{V_D(L) - V_D(L - \kappa L/D)}{V_D(L)} = 1 - e^{-\kappa}$$

and interpret this result (e.g. for small integer values of $\kappa$).

b) Can you think of objects, which **do not** follow the relation between linear dimension and volume given in a)? *[extra point for concrete example]*

c) Next we consider a $D$-dimensional cube with linear dimension, $L$, and a $D$-dimensional sphere with radius $R$. Justify the following relations between volume ($V$) and surface area ($S$)

$$\begin{aligned} \frac{d}{dL} V_D(L) &= \frac{1}{2} S_D(L) \qquad \text{cube} \\ \frac{d}{dR} V_D(R) &= S_D(R) \qquad \text{sphere}\,. \end{aligned} \tag{1}$$

For $V_D(L) = L^D$ we get $S_D(1) = 2D$ for the surface area of the unit-cube. Is this correct? For the sphere we get (from $V_D \sim R^D$): $V_D(R) = \frac{R}{D} S_D(R)$. What does this mean for large $D$?

d) Determine the surface area, $S_D$, of the $D$-dimensional unit sphere by starting with the $D$-dimensional Gauss integral

$$\int\limits_{-\infty}^{\infty} d^D\mathbf{r}\, e^{-\mathbf{r}^2}\,,$$

where $\mathbf{r} = (x_1, \ldots, x_D)^T$. Furthermore, use the fact that for radially symmetric functions $f(x)$

$$\int\limits_{-\infty}^{\infty} d^D\mathbf{r}\, f(|\mathbf{r}|) = S_D \int\limits_{0}^{\infty} dr\, r^{D-1} f(r)\,,$$

where we assume that $f(r)$ decays sufficiently fast such that the integrals exist.
★ See problem I and use the fact that the $D$-dimensional Gaussian integral factorizes.

e) Finally, calculate the volume, $V_D$, of the $D$-dimensional unit sphere explicitly:

$$V_D(1) = \int_{r\leq 1} d^D r$$

★ Use spherical coordinates and that fact that the integration over all angles gives the surface area $S_D$.

## III. MASTER EQUATION FOR RADIOACTIVE DECAY *[(4+4) PTS]*

The decay of radioactive nuclei is given by

$$N(t) = N_0 e^{-\lambda t}$$

where $N(t)$ is the average number of nuclei, which did not decay after time $t$ given that the initial number was $N_0$ at $t = 0$. $\lambda$ is the decay rate and $\lambda\Delta t$ is the probability for a particular nucleus to decay in a (short) time interval $\Delta t$ (Explanation?).

a) Explain the following expression for the probability to find $N$ non-decayed nuclei after time $t$

$$w_N(t) = \binom{N_0}{N} \left(e^{-\lambda t}\right)^N \left(1 - e^{-\lambda t}\right)^{N_0 - N}\,, \quad N = 0, 1, \ldots, N_0\,.$$

Check the normalization of $\sum_N w_N(t)$ and the average $\sum_N N w_N(t)$.

b) Show that $w_N(t)$ satisfies the "master equation"

$$\dot{w}_N(t) = (N+1)\,\lambda w_{N+1}(t) - N\lambda w_N(t)$$

and interpret the result.

## IV. DISTRIBUTIONS *[(2 EACH) PTS]*

Here we consider $N$ particles, which shall be distributed into $g$ containers. Calculate the number of possible distributions for the following cases:

a) all particles are distinguishable (classical) and each container has infinite capacity.

b) all particles are distinguishable and each container holds at most one particle.

c) all particles are identical (indistinguishable) and each container has infinite capacity. ["Bosons"]

d) all particles are identical and each container holds at most one particle. ["Fermions"]

★ In quantum mechanics the containers in cases c) and d) correspond to single particle states.