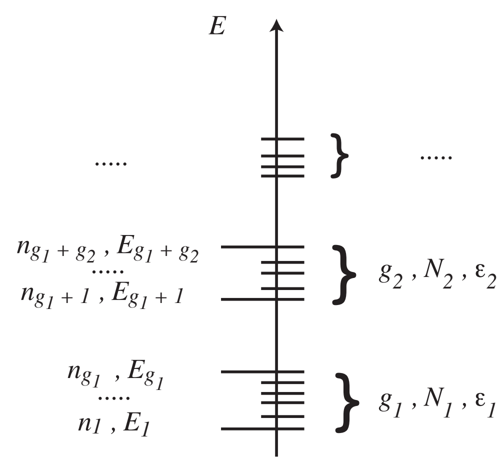
Homework 4 HW

due *2022-03-22*, before class

Exams (tentatively) midterm: **Thursday, March 24, 2022**, *11:00-12:15*
final: **tbd**

XII. BOSE AND FERMI STATISTICS *[(2+6+6+5) PTS] [BONUS: (1+2+2) PTS]*

We consider N non-interacting Bosons (Fermions) having single particle states $|\nu\rangle$ with energies E_ν. *Microstates* are determined by the occupation numbers n_ν of all single particle states, where $n_\nu = 0, 1, 2, \ldots$ for Bosons or $n_\nu = 0, 1$ for Fermions.
In a coarse-grained description we group g_i quasi-degenerate single particle states into groups $i = 1, 2, \ldots$. The particles in each group have energy ϵ_i and the occupation number per group shall be N_i (see figure). For example: $\epsilon_1 \approx E_1 \approx E_2 \approx \ldots \approx E_{g_1}$, $N_1 = n_1 + \ldots + n_{g_1}$; $\epsilon_2 \approx E_{g_1+1} \approx \ldots \approx E_{g_1+g_2}$, $N_2 = n_{g_1+1} + \ldots + n_{g_1+g_2}$.
A *macrostate* is defined by the occupation numbers $M = \{N_i\}$.



Solve each of the following parts for Bosons and Fermions.

a) How many microstates, $W(N_1, N_2, \ldots)$ correspond to a macrostate M? (see lecture) [*Bonus point*: $W(N_1, N_2, \ldots)$ for classical, distinguishable particles.]

b) Now, the total particle number $N = \sum_i N_i$ and energy $E = \sum_i \epsilon_i N_i$ shall be fixed. Find the occupation numbers $\tilde{N}_i$ of the macrostate, which has the largest number of microstates. [*Bonus points*: Do the same for classical, distinguishable particles.]
★ Use the method of Lagrange multipliers as shown in the lecture and use $N_i, g_i \gg 1$ and additionally $g_i - N_i \gg 1$ for Fermions.

c) Approximate $W(M) = W(\tilde{N}_1 + \Delta N_1, \tilde{N}_2 + \Delta N_2, \ldots)$ by expanding $\ln W(M)$ in second order in $\Delta N_1 = N_1 - \tilde{N}_1$. [*Bonus points*: Do the same for classical, distinguishable particles.]

d) Calculate $\langle N_i \rangle$, $\langle \Delta N_i \rangle$, and $\left\langle (\Delta N_i)^2 \right\rangle$ using the approximation found in c). Discuss the connection between $\langle N_i \rangle$ and $\tilde{N}_1$ as well as the magnitude of the fluctuations $\left\langle (\Delta N_i)^2 \right\rangle$.

XIII. CONTINUUM LIMIT *[10 PTS]*

For the calculation of particle fluctuations, we usually replace the summation over particle numbers by integrals, e.g.

$$\left\langle(\Delta N)^2\right\rangle = \frac{\sum_N (\Delta N)^2 e^{\Delta S(N)/k_B}}{\sum_N e^{\Delta S(N)/k_B}} \approx \frac{\int dN\, (\Delta N)^2 e^{\Delta S(N)/k_B}}{\int dN\, e^{\Delta S(N)/k_B}}$$

with $\Delta S(N) \approx -k_B \frac{c}{2}(\Delta N)^2$. Determine under which condition this continuum approximation is valid. Approximate the error by using the Poisson summation formula

$$\sum_{N=-\infty}^{\infty} f(N) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} dN\, f(N) e^{2\pi \imath k N}\,.$$

XIV. ONE-DIMENSIONAL RELATIVISTIC GAS *[(3+3+3+3) PTS]*

Here we consider a non-interacting gas of N relativistic particles in one dimension. The gas is confined in a container of length L, i.e., the coordinate of each particle is limited to $0 \le q_i \le L$. The energy of the i$^{\text{th}}$ particle is given by $\epsilon_i = c|p_i|$.

a) Calculate the single particle partition function $Z_1(T, L)$ for given energy E and particle number N.

b) Calculate the average energy E_1 and the heat capacity C_1 per particle from $Z_1(T, L)$.

c) Calculate the Boltzmann entropy $S_B(E, N)$ of all N particles. Consider them as indistinguishable.
★ Use the definition of the free energy.

d) Show that the Boltzmann entropy is extensive in the limit of large particle numbers.

★ Use Stirling's formula for large N.

XV. PROJECTION OPERATORS *[1 PT EACH]*

Projection operators, $\hat{J}_M$, play an important role in defining macrostates for quantum systems. Let us assume that the common eigenstates $|X_\nu\rangle$ of macro-operators span an orthonormal basis of the Hilbert space, i.e., $\langle X_\nu | X_\mu\rangle = \delta_{\nu,\mu}$ and $\sum_\nu |X_\nu\rangle\langle X_\nu| = \hat{1}$. With this, macrostates, M, are defined as disjoint groups ($\nu \in M$) whose projection operators are

$$\hat{J}_M = \sum_{\nu \in M} |X_\nu\rangle\langle X_\nu|\,.$$

Prove that the $\hat{J}_M$ form a complete set of projection operators:

a) $\sum_M \hat{J}_M = \hat{1}$

b) $\hat{J}_M \hat{J}_{M'} = \delta_{M,M'} \hat{J}_M$

c) all eigenvalues of $\hat{J}_M$ are either 0 or 1.

For Bosons and Fermions the symmetrization ($\hat{J}_+$) and anti-symmetrization ($\hat{J}_-$) operators are defined as

$$\hat{J}_\pm = \frac{1}{N!}\sum_P (\pm 1)^{\chi(P)} \hat{P}\,,$$

where the $\hat{P}$ are permutation operators (which permute the particle coordinates) and $\chi(P)$ is the number of transpositions in permutation P. Show that:

d) $\hat{J}_\pm^2 = \hat{J}_\pm$

e) $\hat{J}_+ \hat{J}_- = 0$

f) $[\hat{J}_\pm, \hat{O}] = 0$ for all observables with $[\hat{P}, \hat{O}] = 0$, $\forall P$.

XVI. PRESSURE EXCHANGE *[6 PTS]*

Consider N particles having the total energy E in a container of volume V. The container is split by a freely moving piston into two chambers $i = 1, 2$ with variable volumes V_i ($V = V_1 + V_2$). Each chamber has a fixed number of particles N_i ($N = N_1 + N_2$). We assume that energy exchange between the two chambers is possible. Show that not only the temperature, but also the pressure is equal in both chambers in thermal equilibrium. Assume that the entropy in each chamber is given by a known function $S_B^{(i)}(N_i, E_i, V_i)$.