Homework HW 3

due *2019-02-28*, before class

Exams (tentatively)	midterm (tentatively):	**Thursday, March 21, 2019**, *9:30-10:45*
	final:	**Thursday, May 9, 2019**

VII. MAXWELL DISTRIBUTION *[(4+2) PTS]*

The Maxwell velocity distribution describes the probability density for the velocity $\mathbf{v} = (v_x, v_y, v_z)$ of a particle with mass m in an ideal gas at temperature T. It is given by

$$N(\mathbf{v})d^3v = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{m\mathbf{v}^2}{2k_B T}\right) d^3v\,.$$

a) Calculate the average of the magnitude of the velocity (speed) $|\mathbf{v}|$.

b) Calculate the probability distribution $N_x(v_x)dv_x$ for the x-component, v_x, of the velocity $\mathbf{v}$.
★ Use $\int_0^\infty dx\, x^n e^{-ax^2} = \Gamma((n+1)/2)/(2a^{(n+1)/2})$ for $a > 0$.

VIII. HARMONIC OSCILLATORS *[4 PTS]*

We consider a system of N uncoupled classical oscillators

$$H(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^{N} \left(\frac{\mathbf{p}_i^2}{2m_i} + \frac{m_i\omega_i^2}{2}\mathbf{q}_i^2\right)\,.$$

Each point $\vec{X} = (\mathbf{q}, \mathbf{p}) \equiv (\mathbf{q}_1, \ldots, \mathbf{q}_N, \mathbf{p}_1, \ldots, \mathbf{p}_N)$ in phase-space represents a microstate of the system. A macrostate $M(E_1, E_2)$ shall consist of all $\vec{X}$ with $E_1 \le H(\vec{X}) \le E_2$. Calculate the phase-space volume $|\Gamma[M(E_1, E_2)]|$. Discuss how the phase-space volume of a shell $M((1-\epsilon)E, E)$ with relative thickness $0 < \epsilon < 1$ compares to the phase-space volume of an ellipsoid $M(0, E)$ in the limit for large N.

IX. BOLTZMANN ENTROPY *[(5+5) PTS]*

Let us consider a system consisting of three sub-volumes with N Ising spins, $s_{i,L} = \pm 1$, in each ($i = 1, \ldots, N$, $L = 1, 2, 3$). Here we neglect their interaction. The macrostate, $M = (m_1, m_2, m_3)$, shall be defined by the total spins $m_L = \sum_{i=1}^{N} s_{i,L}$ of each sub-volume.

a) Calculate the Boltzmann entropy $S(M)$. Using this, obtain the equilibrium state $\tilde{M} = (\tilde{m}_1, \tilde{m}_2, \tilde{m}_3)$ and its entropy $S(\tilde{M})$.
★ Use Stirling's formula.

b) Calculate the average quadratic fluctuations $\langle m_1^2 \rangle$ of the total spin in sub-volume 1.
★ Expand $S(M)$ up to order m_1^2.

X. POTTS MODEL *[(5+7) PTS]*

In the classical 3-state Potts model, each of the N "spins" can have three states. A microstate is defined by the states (1, 2, or 3) for all spins. A macrostate, on the other hand, is defined by the occupation number $N_i = \frac{N}{3} + m_i$ $(i = 1, 2, 3)$ of the three spin states. The m_i obviously have to fulfill the condition $m_1 + m_2 + m_3 = 0$.

a) Find the number $W(m_1, m_2)$ of microstates belonging to a macrostate defined by $m_{1,2}$. What is the total number of microstates?

b) For which $m^0_{1,2}$ is $W(m_1, m_2)$ maximal?
★ Expand $\ln W(m_1, m_2)$ around m^0_1, m^0_2 up to order $(m_i - m^0_i)^2$. How large are the deviations of the macro-variables $m_{1,2}$?

XI. DIRECTED POLYMER II *[(4+4+4) PTS]*

Here we consider a simplified model for a chain-molecule. We assume that the atoms, $i = 0, 1, \ldots, N$, of the chain sit on grid points of a square lattice (x_i, y_i) (x_i, y_i are integers). Furthermore, the chain starts at $(x_0, y_0) = (0, 0)$ and the following positions are constrained by $x_i - x_{i-1} = 1$ and $|y_i - y_{i-1}| = 1$. Therefore the chain is directed along x-direction, avoiding, e.g., loops. The positions of the N atoms define the microstate.

a) In problem VI we defined as the macrostate (N, y) all microstates, where the chain ends at $y_N = y$. Here we define, similarity, a macrostate as (i, y), where the position $y_i = y$ of the i-th atom is fixed. What is the number, $W(i, y)$, of the corresponding microstates? Calculate also: $\sum_y W(i, y)$.
★ Remember, there are $N - i$ more atoms in the chain.

b) Next, we define the macrosate as $(i, y; N, y')$, where the position of the i-th atom is $y_i = y$ **and** the position of the last atom is $y_N = y'$. What is the number, $W(i, y; N, y')$, of the corresponding microstates? Calculate also: $\sum_{y,y'} W(i, y; N, y')$.

c) Finally, let us assume that a detector can only measure the position of every 10th atom, i.e., only $y_{10}, y_{20}, \ldots, y_N$ are detected (assume $N = 10n$, $n \in \mathbb{N}$). How many microstates correspond to each experimentally measured "macrostate", $W(y_{10}, y_{20}, \ldots, y_N)$? Calculate also $\sum_{y_{10}, y_{20}, \ldots, y_N} W(y_{10}, y_{20}, \ldots, y_N)$.