

Homework

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HW

due 2019-02-12, before class

Exams (tentatively)

midterm (tentatively): **Thursday, March 21, 2019, 9:30-10:45**  
 final: **Thursday, May 9, 2019**

V. CENTRAL LIMIT THEOREM [(2+2+3+4) PTS]

Let  $x$  be a random variable with (probability) distribution function  $w(x)$ , which is positive,  $w(x) \geq 0$ , and normalized  $\int_{-\infty}^{\infty} w(x) dx = 1$ . The average,  $\bar{x}$ , and deviation,  $(\Delta x)^2$ , are defined as

$$\bar{x} \equiv \int_{-\infty}^{\infty} x w(x) dx$$

$$(\Delta x)^2 \equiv \int_{-\infty}^{\infty} (x - \bar{x})^2 w(x) dx = \overline{x^2} - \bar{x}^2.$$

We assume that both these quantities exist (for, e.g.,  $w(x) = (1 + x^2)^{-1}/\pi$  the above integrals do not converge).

- a) Let us assume that the random numbers are the result of a measurement of one observable. It makes sense to calculate the arithmetic average,  $y = \frac{1}{N} \sum_{i=1}^N x_i$ , of  $N$  measurements,  $x_i$ . We can now assume that  $y$  is another random variable with the distribution function  $w_N(y)$ . The latter is obtained by summation of the probabilities  $w(x_1) \cdots w(x_N) dx_1 \cdots dx_N$  with the condition that  $y = \frac{1}{N} \sum_{i=1}^N x_i$ , i.e.

$$w_N(y) = \int dx_1 \cdots dx_N w(x_1) \cdots w(x_N) \delta\left(y - \frac{1}{N} \sum_{i=1}^N x_i\right).$$

Calculate the average and the deviation of  $y$  and discuss the results.

- b) Calculate the average and the deviation of

$$u \equiv \frac{1}{\sqrt{N} \Delta x} \sum_{i=1}^N (x_i - \bar{x}).$$

- c) The Fourier transform  $\tilde{w}(t) \equiv \int e^{itx} w(x) dx = \overline{e^{itx}}$  of the distribution function  $w(x)$  is a generating function of the "moments"  $\overline{x^\kappa} = \int x^\kappa w(x) dx$  of the distribution, in the following sense:

$$\overline{x^\kappa} = \left(\frac{1}{i} \frac{d}{dt}\right)^\kappa \tilde{w}(t)|_{t=0}, \quad \kappa = 1, 2, 3, \dots$$

Show this identity. Conversely, these moments also define the distribution function  $w(x)$ . Find the distribution function for the moments defined as  $\overline{x^\kappa} = \kappa! a^{-\kappa}$  for even  $\kappa$  and  $\overline{x^\kappa} = 0$  for odd  $\kappa$  ( $a$  is a positive constant).

★ Use the integral  $\frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{-itx} \frac{a^2}{a^2+t^2} = \frac{a}{2} e^{-a|x|}$  (which can be calculated using residues).

- d) The central limit theorem states that – under rather general conditions for  $w(x)$  – the quantity  $u$  [see b)] follows the Gaussian (or normal) distribution

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}$$

for  $N \rightarrow \infty$ . To prove this, use the generating function  $\tilde{w}_N(t) = \int e^{itu} w_N(u) du$  [ $u$  and the corresponding  $w_N(u)$  as in b) ]. Show that  $\tilde{w}_N(t) \propto \left( \tilde{w}(t/(\Delta x \sqrt{N})) \right)^N$ , where  $\tilde{w}(t)$  is the Fourier transform of  $w(x)$ . Then expand,  $\ln \tilde{w}_N(t)$ , to second order in  $t$ . The normal distribution follows from the inverse Fourier transform.

## VI. DIRECTED POLYMER [(2+3) PTS]

Let us consider a square grid, which has been rotated by  $45^{\text{deg}}$  (see figure). On this grid we define a path, the directed polymer, which starts at the origin ( $t = 0$ ) and extends in the positive  $t$ -direction (at each grid point the path goes either left or right; and steps in the negative  $t$ -direction are not permitted).

All paths of length  $N$  end at the same  $t$ -position (depicted by the  $m$ -axis in the figure). The distance of the endpoint to the  $t$ -axis is characterized by a number  $m$ . Each path represents a microstate and the endpoint (i.e.  $m$ ) the macrostate.

- Find the number of microstates,  $W(m)$ , for each macrostate,  $m$ . How many microstates exist for a directed polymer of length  $N$ ?
- What is the value  $m_0$  of  $m$  for which  $W(m)$  is maximal? Expand  $\ln(W(m))$  around  $m_0$  ( $m = m_0 + \Delta m$ ) to second order in  $\Delta m$ .

*Hint:* Assume that the grid points of rotated grid have integer  $m$  coordinates and that  $N + m$  is even.

