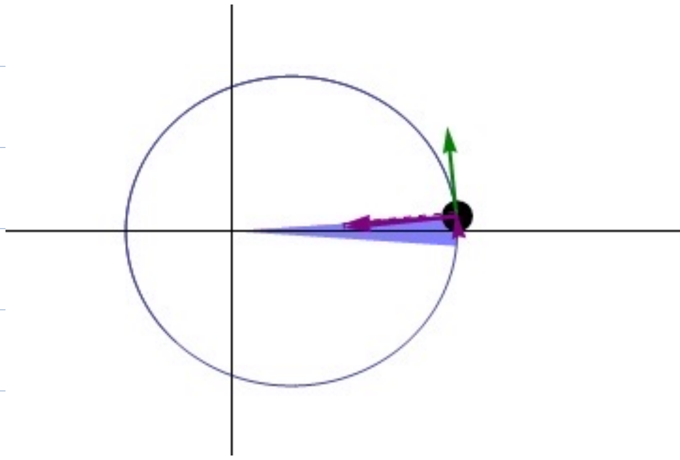
Second law of Kepler:

A line joining a planet and the sun sweeps out equal areas during equal intervals of time.



blue area is constant

green: velocity

purple: acceleration

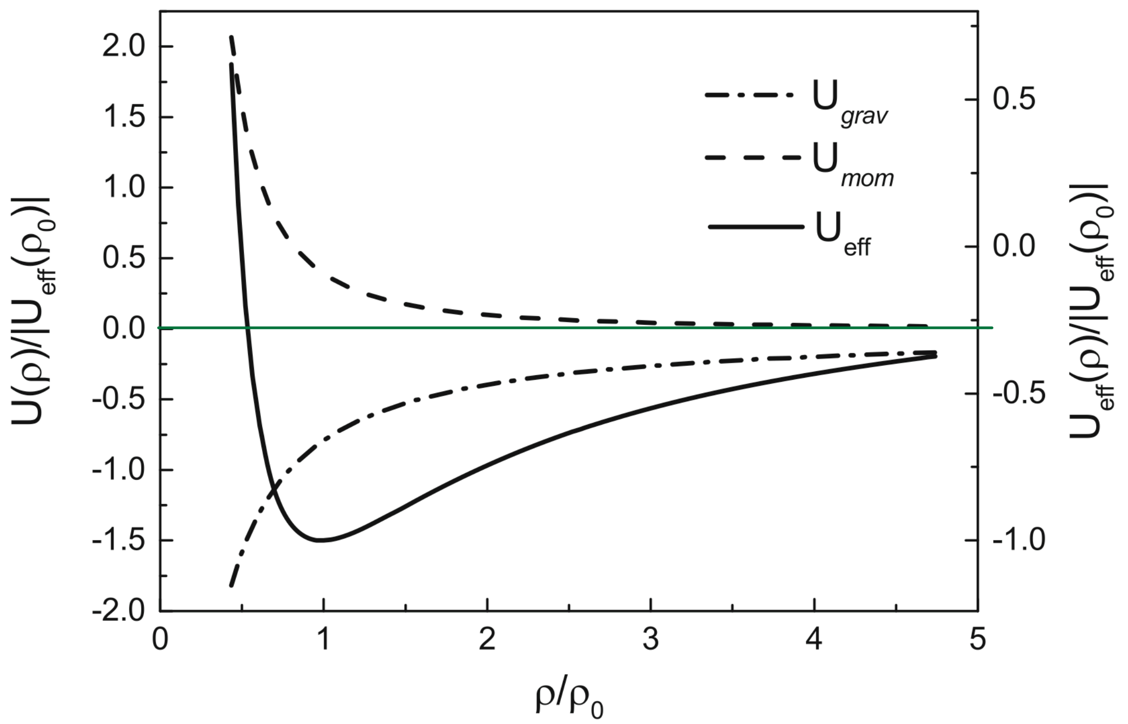
$$\dot{\varphi} = \frac{L}{m\varrho^2}$$, L: angular momentum = constant

m: reduced mass

radial velocity: $\dot{\varrho} = \pm\sqrt{\frac{2}{m}\left(E - U_{eff}(\varrho)\right)}$

$$\Rightarrow \quad \frac{1}{2} m \dot{\varrho}^2 = E - U_{eff}(\varrho) \quad , \quad E = \text{const}$$

$$U_{eff} = -\frac{\alpha}{\rho} + \frac{L^2}{2m\rho^2} \quad , \quad \alpha = \text{const.}$$



Define : $a \equiv \frac{L^2}{m \alpha \rho_0}$, $\rho_0 = \rho(t_0)$ initial value

$$e \equiv \sqrt{1 + \frac{2EL^2}{m\alpha^2}} \quad \text{eccentricity}$$

$$\Rightarrow e^2 - 1 = \frac{2EL^2}{m\alpha^2} \equiv \tilde{E}$$

dimensionality $[\rho_0]$ = length

$[\alpha]$ = energy × length

$[L]$ = energy × time

⇝ a and e are dimensionless

Introduce dimensionless length and time

$$\tilde{\rho} = \rho / \rho_0$$

$$\tilde{t} = \frac{\alpha}{L\rho_0} t$$

Thus : $\boxed{\frac{d\varphi}{d\tilde{t}} = \frac{a}{\tilde{\rho}^2}}$ with $\tilde{\rho}_0 = 1$

$$\boxed{\frac{d\tilde{\rho}}{d\tilde{t}}} = \pm \frac{L\rho_0}{\alpha \rho_0} \sqrt{\frac{2}{m}\left(E + \frac{\alpha}{\rho} - \frac{L^2}{2m\rho^2}\right)}$$

$$= \pm \left[\frac{2L^2}{m\alpha^2}\left(E + \frac{\alpha}{\rho} - \frac{L^2}{2m\rho^2}\right)\right]^{1/2}$$

$$\boxed{= \pm \left[e^2 - 1 + \frac{2a}{\tilde{\rho}} - \frac{a^2}{\tilde{\rho}^2}\right]^{1/2}}$$

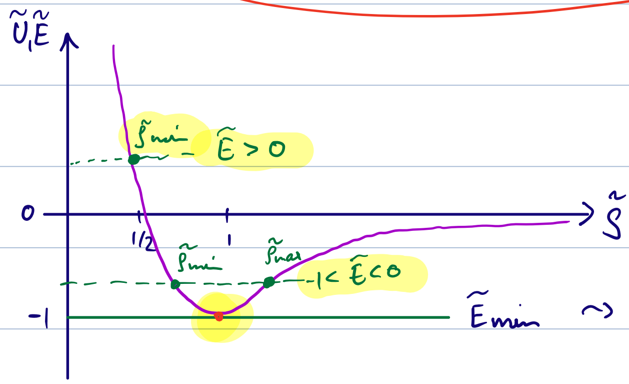
Explicit Euler scheme : $\varphi_{n+1} = \varphi_n + h \dfrac{a}{\tilde{\rho}_n^2}$, $\varphi_n = \varphi(t_n)$

$\tilde{\rho}_n = \tilde{\rho}(t_n)$

$\tilde{t}_n = \tilde{t}_0 + nh$

choose "+" : $\tilde{\rho}_{n+1} = \tilde{\rho}_n + h\left(e^2 - 1 + 2\dfrac{a}{\tilde{\rho}_n} - \dfrac{a^2}{\tilde{\rho}_n^2}\right)^{1/2}$

potentials :



$\rightsquigarrow \rho_0 = \kappa$

$\tilde{\rho}_0 = 1$

For $-1 < \tilde{E} < 0 \Leftrightarrow 0 < e < 1$ $\quad \tilde{\rho}_{min} = + \frac{a}{\tilde{E}}\left(1 - \sqrt{1+\tilde{E}}\right)$

$\tilde{\rho}_{max} = + \frac{a}{\tilde{E}}\left(1 + \sqrt{1+\tilde{E}}\right)$

for $\tilde{E} > 0$ $\quad \rho_{min} = \frac{a}{E}\left(\sqrt{1+E} - 1\right)$

for $\tilde{E} = 0$ $\quad \rho_{min} = \rho_{max} = \frac{a}{2}$

- Make sure that $E \geq U_{eff}(\rho)$

terminate for $U_{eff} > E$ or change sign

in $\dot{\rho}$ equation